Abstract

We study the performance of all-pay mechanisms for the private provision of a public good in a simple setting in which donors compete for a prize of commonly known value. Within the set of voluntary participation mechanisms, the lowest-price all-pay auction, in which the highest bidder wins the prize but all bidders pay the lowest bid submitted, is the optimal mechanism. The highest amount for the public good is generated in the unique, symmetric, mixed-strategy equilibrium of this mechanism. We derive the equilibrium distribution function in a closed form for any number of bidders. The lowest-price all-pay auction has, however, asymmetric equilibria resulting in zero total donations. We compare the performance of lowest-price and the own-price auctions and lotteries in lieu of voluntary contributions with a battery of laboratory experiments. The performance of the optimal mechanism depends on the level of competition. The lowest-price all-pay auction dominates the remaining auction formats with three competing bidders, but is inferior to the own-pay auction and the lottery with only two bidders.

JEL Classification: D44, D64

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"The safety net will be stretched thin in some places and eliminated entirely in others. For the functions government no longer will be able to provide, we must turn to neighbors, private charities, faith-based organizations, and other local programs. Our communities, more than ever, will be asked to step up."

— Chris Gregoire, Washington state governor, December 15, 2010

1 Introduction

With the steady growth of the philanthropic market and the decline of government funding for various areas of public life (e.g. arts, culture, public media, higher education, hospital services, environmental protection, etc.), the design of mechanisms for the private provision of public goods has significantly gained in importance. It is well-known that fund-raising efforts based on voluntary contributions cannot provide public goods at their socially optimal levels because they suffer from a free-rider problem. While the theoretical literature on the efficient implementation of public goods suggested numerous remedies to the free-rider problem, most of the solutions rely on taxation and subsidy schemes to counterbalance free-riding incentives (Clarke, 1971; Groves, 1973; Groves and Ledyard, 1977; Walker, 1981). These schemes are typically outside the domain of private charitable organizations which can use rewards, but not punishment or coercion to ensure that individuals contribute.

Several recent articles establish a theoretical ranking of fund-raising mechanisms based on rewards only, and evaluate their fund-raising potential relative to the voluntary contribution mechanism. Morgan (2000) and Morgan and Sefton (2000) present theoretical results and experimental evidence showing that lotteries generate higher revenue than the voluntary contribution mechanism. Goeree, Maasland, Onderstal and Turner (2005) study the optimal design of fund-raising mechanisms in the symmetric independent private value model in which bidders receive an additional benefit proportional to the revenue generated by the mechanism. In this setting, the revenue equivalence theorem does not hold. Goeree et al. (2005) show that all-pay auctions generate more revenue than winner-pay auctions. The optimal mechanism in this model is the lowest-price all-pay auction – a mechanism in which the highest bidder wins but all bidders pay the lowest bid – augmented by an appropriately chosen entry fee and a reserve price which serve the purpose of screening out low-value bidders. The optimality of this auction, as the authors explain, results from the combination of two design features: (a) the auction allocates the prize efficiently (the highest value bidder wins with certainty), and (b) when winning the prize, each bidder retains the positive externality arising from the contributions of the other bidders. Goeree et al. (2005) suggest that, given that fund-raisers often rely on lotteries – and lotteries are all-pay mechanisms – a switch to the lowest-price all-pay auction would further enhance charitable giving because it would correct the inefficiency associated with the allocation of the prize. And, based on this argument, Goeree et al. (2005) conjectured that the all-pay auction formats “[…] are easy to implement and may
revolutionize the way in which money is raised.”

Given the lack of evidence from naturally occurring environments, it is important to understand how the lowest-price all-pay auction performs, both from a theoretical and a behavioral standpoint, in alternative settings. Does the lowest-price all-pay auction outperform lotteries even when efficiency is not an issue? Further, while mechanism design theory has been a powerful approach for constructing optimal mechanisms in both complete and incomplete information settings, one shortcoming of this approach that is often pointed out, is that the constructed optimal mechanisms might have multiple equilibria, some of which with very undesirable properties. What is the set of all equilibria of the lowest-price all-pay auction and how much revenue is generated in each equilibrium? How does this newly engineered auction perform, theoretically and in the laboratory, in comparison to the more customary all-pay formats, e.g. lotteries and own-bid all-pay auctions? And how does the number of participants and other variables such as the size of the prize and the marginal per capita return affect revenue? Filling these gaps in our knowledge, in addition to being of direct interest for the development of auction theory, might also have practical value to leaders in the field who are looking for new strategies to reach funding targets. Further, examining the limits to fund-raising through private contributions might also help improve policy predictions (Andreoni, 1998). As tax revenues decline, budget deficits grow, and cuts in social programs are being discussed, policymakers try to assess to what extent communities can continue to provide public goods without raising taxes.

This paper contributes to the research program which explores the limits of the private provision of public goods with voluntary participation mechanisms. We study optimal fund-raising mechanisms in the simplest possible environment, in which a prize of commonly known value \( V \) is awarded to one of the contributors. A mechanism in this simple setting consists of an allocation rule which maps the willingness to contribute (the bids) of all participants into a probability of winning the prize for each participant, and a payment rule which specifies how the agents’ willingness to contribute translates into actual donations. In our setting there are \( n \geq 2 \) participants with a given budget \( B \) who decide how much of it to contribute to a public good. Every participant receives a benefit of \( \alpha \) for each dollar raised for the public good. Thus, each donor derives utility from three sources: the chance of winning the prize, the funds raised for the public good, and the amount left for private consumption.

When \( \alpha \geq 1/n \), voluntary participation mechanisms exist which can implement the Pareto efficient outcome even when no prize is allocated. Andreoni (1998) and Bagnoli and Lipman (1989) show how such mechanisms can be constructed theoretically, and Bagnoli and McKee (1991) provide evidence that these mechanisms perform well in laboratory experiments. More recently, Orzen (2008) reports laboratory experiments with a mechanism which he termed the “lowest common denominator.” This mechanism, in which all participants pay the lowest contribution allowed, generates efficient provision of the public good both theoretically and in the experimental lab. In this mechanism, donating their entire budget is a dominant strategy.
for each participant.

This paper focuses on the more problematic, yet more interesting and probably more realistic case in which the collective benefit to the donors from the public good is lower than the total cost of provision \((\alpha < 1/n)\). Such a scenario seems to apply often in reality because, typically, fund-raising efforts reach only a subset of the beneficiaries from the public good. Thus, the joint utility of the actual donors might often be below the total cost for public good provision even though the community benefits of the public good exceeds total cost. In the case of charitable giving this scenario is even more clear cut because the revenues generated by donors go to the benefit of others. This scenario is of theoretical interest also because mechanisms that produce the efficient outcome do not exist. In this paper we explore which mechanism comes closest to efficiency.

We show that, theoretically, the lowest price all-pay auction – with no entry fee and no reserve price – is optimal. It generates the highest expected revenue among all mechanisms in which potential donors participate voluntarily. This expected revenue equals \(V/(1 - \alpha \cdot n)\) and is attained when bidders play symmetric strategies. We show that the symmetric equilibrium, given in mixed strategies, is unique, and we characterize the equilibrium cumulative distribution function in closed form for any number of bidders. This auction, however, supports asymmetric equilibria in which one of the participants does not donate and total donations equal zero. For high enough budget \(B\), this equilibrium always exists.\(^1\) While the optimal mechanism we derive here is similar in structure to the optimal mechanism obtained in Goeree et al. (2005) for the symmetric independent private value model, some important differences exist. In our setting the optimal mechanism entails neither an entry fee nor a reserve price; in equilibrium the prize is allocated with a probability of one. The use of the reserve price and the entry fee in Goeree et al.’s incomplete information setting serve the purpose of screening out the low valuation bidders who, by submitting low bids, suppress revenues in the lowest-price all-pay auction. In the complete information case in which all bidders have the same common value for the auction item, we show that the optimal mechanism entails neither entry fees nor reserve prices. It may appear at first that our results can be obtained as a limiting case of the independent private value model in which the distribution of bidder valuations in the limit reduces to a single point. While such a conjecture may seems intuitively appealing, it cannot be pursued with the use of equilibrium purification arguments (see e.g. Govindan, Reny and Robson, 2003; Harsanyi 1973) because the available equilibrium purification results apply to finite games only. In fact, the equilibrium distribution functions we obtain do not seem to be special cases of Goeree et al.’s (2005) Bayes-Nash equilibrium distributions when uncertainty vanishes.

We tested the validity of our theoretical predictions with a series of laboratory experiments in which two payment rules – pay your own bid, and pay the lowest bid – are combined

\(^1\)The existence of these zero-contribution equilibria is not tied to the particular complete information setting considered here; these equilibria also exist in the incomplete information private value setting.
with three rules for allocating the prize – auction, lottery and random assignment of the prize regardless of individual contributions. In our experiments there are four players with a marginal per capita return of $\alpha = 0.3$, yet only two or three of them have a budget ($B = 100$) and are able to donate (we call them “active” players). The remaining participants derive benefit only from the overall provision of the public good (“passive” players). The active players compete for a prize of $V = 20$ by donating in one of the six described mechanisms. We compare the performance of the six different mechanisms and examine how incentives provided affect the bidding behavior of donors.

We find several important differences between theoretical predictions and laboratory behavior. The first, and probably most important finding, is that the relative performance of the lowest-bid all-pay auction critically depends on the number of donors. Among all mechanisms, the theoretically optimal format generates the smallest revenue in the lab with $n = 2$ bidders and the greatest revenue with $n = 3$ bidders, whereby the differences with some other treatments are statistically significant. Thus, a switch from a lottery to this auction format seems beneficial only if the fund raising organization elicits small contributions from a sufficiently large number of donors. The second discrepancy concerns the performance of the own-bid all-pay auction. Somewhat counter-intuitively, theory suggests that the total amount of donations is invariant to the number of bidders. Even more surprising, we find in the experiment that total revenues are significantly smaller with $n = 3$ donors than with $n = 2$ donors. Thus, the increased competition for the prize actually lowers total donations. Finally, donated amounts in the experiment are substantially higher than those predicted by theory. In line with theory, we find that total amount raised for the public good increases in the number of bidders in the lottery and in the lowest auction format.

2 Related literature

2.1 Prize based mechanisms

The performance of prize-based mechanisms for fund-raising has been a subject of much theoretical and experimental research in recent years. Most closely related to the present study are the papers by Orzen (2008) and Duffy and Matros (2010) who compare the revenue-generating properties of all-pay auctions and lotteries in the same complete information framework analyzed here. Both contributions derive the symmetric mixed strategy equilibrium of the own-price all-pay auction with a budget constraint. Complementing these efforts, we show that the underlying public good game can be transformed in a standard auction without a public good element, but with an inflated prize. This observation allows us to combine the theoretical insights from the literature on the all-pay auction with complete information (e.g. Baye, Kovenock and de Vries, 1996), and on political lobbying with caps (Che and Gale, 1998) and provide a complete characterization of all equilibria of the own-bid all-pay auction. We find that, in the case of two bidders, the symmetric mixed strategy equilibrium is the only
equilibrium of the game. With three or more bidders, the symmetric equilibrium is unique, but there are additional asymmetric equilibria. If the budget is sufficiently large relative to the prize and the marginal per capita return, all equilibria generate the same level of expected contributions equal to $V/(1 - \alpha)$ independent of the number of bidders.

Orzen (2008) is, to the best of our knowledge, the only experimental study of bidding in the lowest-price all-pay auction. In Orzen’s experimental design $\alpha \cdot n > 1$, and in this environment any arrangement which ask a bidder to donate a certain amount, provided that all other bidders agree to donate the same amount, is sustainable even when no prize is allocated. In particular, contributing the entire budget is a weakly dominant strategy for each player; even if no prize is allocated – a mechanism which Orzen termed “the lowest common denominator.” Conducting experiments with four bidders, Orzen finds that the lowest-price formats, the lowest-price auction and the lowest common denominator, outperform the lottery and the own-bid all-pay auction. In particular, Orzen finds that in the last round of the experiment all subjects contribute their entire budget in both lowest-price formats – a striking confirmation of the theoretical prediction. Major difference between Orzen’s setting and ours is that we put potential donors in an environment that is susceptible to the free-riding problem. In our setting $\alpha \cdot n < 1$ and the lowest common denominator cannot raise any positive amount of donations in this environment. Further, donating the entire budget in the lowest-price all-pay auction is not an equilibrium behavior either. We show, however, that, in this setting, the lowest-price all-pay auction generates the largest amount of donations that any voluntary mechanism can generate, and derive the equilibria of this auction in detail. We find that the lowest-price all-pay auction is very sensitive to the number of active bidders as the revenues in the case of three bidders are substantially higher than the revenues in the two bidder case.

Duffy and Matros (2010) compare revenues in the own-price all-pay auction and the lottery while vary the number of bidders and the marginal per capita return. Considering treatments with two and ten bidders, they report that individual contributions decline with the increase in the number of bidders, but total donations increase. More importantly, they find no significant difference in total donations across the two mechanisms – a finding which may explain the prevalence of lotteries.

Experimental tests of all-pay mechanisms have been conducted in several alternative settings. Corazzini, Faravelli and Stanca (2010) find that the lottery outperforms the own-bid all-pay auction in an incomplete information setting in which donors vary by income. They also find that prize based mechanisms perform better than voluntary contributions. Schram and Onderstal (2009) conduct an experimental study in independent private value model considered in Goeree et al. (2005) and observe, on the contrary, that the own-bid all-pay auction performs better than the lottery. Lange, List and Price (2007) compare single-prize

2It is common that the set of donors is only a small subset of all beneficiaries from the public good. It is, thus plausible to assume that the joint benefit only from the participants in the fund-raising event might be below the total benefit that donors derive from private consumption.
and multiple-prize lotteries in lieu of the voluntary contribution mechanism in settings with different risk attitudes and marginal per capita returns. The lotteries outperform the voluntary contribution mechanism, yet the optimal fund-raising lottery design is sensitive to the risk attitudes of contributors and the composition of their marginal per capita returns. In a door-to-door fund-raising field experiment in which 5000 households are asked to support the Center for Natural Hazard Mitigation Research at East Carolina University, Landry, Lange, List, Price and Rupp (2006), find that lotteries raise more funds than the voluntary contribution mechanism, controlling for various factors related to the solicitor. Carpenter, Holmes, and Matthews (2008) compare the performance of the own-bid all-pay auction and winner-pay auctions (first and second price) in a field experiment conducted during the fund-raising festival of four preschools in Vermont in the spring of 2003. Contrary to the recent theoretical results, they find that the own-bid all-pay auction is dominated by the first-price auction – a finding that the authors attribute to endogenous participation, and the lack of familiarity with the all-pay format.

2.2 Other mechanisms

Laboratory and field experiments have been conducted with other incentive schemes that do not rely on a prize.

Matching and rebates. Both in within-subject and between-subject laboratory experiments, Eckel and Grossman (2003, 2006) find that subjects donate more when a matching grant (providing additional funds proportionally related to the amount contributed by the givers) is offered compared to a functionally equivalent rebate (giving a portion of the donation back to the contributor). In a field experiment using direct mail solicitations, Karlan and List (2007) find that matching increases both total donations and the chances of respondents to donate, but the size of the match, whether being a dollar, two, or three dollars for each dollar donated, have no additional effect. List and Lucking-Reiley (2002) compare the effectiveness of seed money and refunds to raise funds from donors in a field experiment for funding the Center for Environmental Policy Analysis in central Florida and conclude that both methods significantly increase contributions.

Point provision mechanism. Another mechanism of practical importance is the point provision mechanism in which the public good is provided only if total contributions meet or exceed a natural or artificially imposed threshold. In a complete information setting, Bagnoli and Lipman (1989) and Bagnoli and McKee (1991) show that, if the total value of the public good to all donors exceeds the cost of provision, the point provision mechanism can implement the efficient allocation in a Nash equilibrium and observe such a behavior in the laboratory experiments they conducted. Rondeau, Schulze and Poe (1999) conduct experiments with the point provision mechanism with a proportional rebate of excess contributions. They find
that subjects reveal their true demand when the group of subjects is large and subjects have heterogeneous valuations. Croson and Marks (1999) consider the provision point mechanism in settings with homogeneous and heterogeneous players regarding the valuation of the public good. There are no significant differences in group contributions across the two treatments, yet groups with heterogeneous valuations exhibit more variance within a group over time, and between groups. Marks, Schansberg, and Croson (1999) introduce suggested contributions into the provision point mechanism, and observe that suggested amounts work well in heterogeneous groups in raising total donations, but are ineffective, or even damaging when used in homogeneous groups. More recently, using data from laboratory and the field, Rondeau, Poe and Schulze (2005) establish that the point provision mechanism performs better than simply asking for voluntary contributions.

**Seed money and refunds.** While the point provision mechanism has Pareto efficient equilibria, there exist also a Nash equilibrium with zero donations (see e.g. Bagnoli and Lipman, 1989). One way to eliminate this equilibrium is through seed money, which reduce the remaining amount that needs to be raised to reach the provision point (Andreoni, 1998). An alternative instrument to stimulate giving is the promise to return the donations of individual donors if the provision point is not met (Bagnoli and Lipman, 1989). List and Lucking-Reiley (2002) report results from a field experiment in which they find significant effects both on participation rates and average contribution in the two mechanisms. Seed money have, however, a larger effect on donations.

### 3 Preliminaries

We consider a set \( N = \{1, \ldots, n\} \) of two or more participants in a fund-raising event. All participants are risk neutral and have the same budget \( B \) which they can allocate between contribution toward the provision of a public good and personal consumption. As in Goeree et al. (2005) the value of the public good for each participant is constant fraction, \( \alpha \), of total funds raised for the public good. If \( \alpha \geq 1/n \), the free-rider problem can easily be resolved with simple arrangements according to which all contributors agree to donate a certain amount, and if at least one of them does not contribute, the public good is not provided (Bagnoli and Lipman, 1989; Bagnoli and McKee, 1991; Orzen, 2008). We, therefore, as in Goeree et al. (2005), consider the case \( \alpha < 1/n \) in which these simple voluntary arrangements for public good provision are not sustainable in equilibrium. In reality, the set of potential beneficiaries of the public good well exceeds the original set of donors, and, in fact, the beneficiaries of charitable donations are often outside the set of the donors. The charitable organization uses a mechanism to award a prize of common value \( V \) to one of the participating donors.
3.1 Mechanisms

We consider a one-shot setting in which each participant \( i \in N \) announces her willingness to contribute to the public good, \( x_i \). A mechanism in our setting consists of an allocation rule \( P_i(x_i, x_{-i}) \) which maps the willingness to contribute of all participants into a probability of winning the prize for each donor, and a payment rule \( C_i(x_i, x_{-i}) \) which specifies how the agents’ willingness to contribute translates into actual donations. If bidders participate in the mechanism with an allocation rule \( P_i(x_i, x_{-i}) \) and a payment rule \( C_i(x_i, x_{-i}) \) the expected payoff of bidder \( i \in N \) is given by

\[
\Pi_i(x_i, x_{-i}) = B - C_i(x_i, x_{-i}) + \alpha \cdot \sum_{j=1}^{n} C_i(x_i, x_{-i}) + P_i(x_i, x_{-i}) \cdot V.
\]

We focus on the set of voluntary mechanisms, that is, mechanisms in which agents cannot be asked to contribute more than their expressed desire to contribute, \( C_i(x_i, x_{-i}) \leq x_i \), and in which the utility from participation (in equilibrium) is not below the utility from non-participation, or donating zero. In our experiments, we focus on three versions of \( P_i \) and two versions of \( C_i \):

\[
P_i(x) = \begin{cases}
\frac{1}{n} & \text{(VCM)} \\
\frac{1}{\text{argmax}_{j \in N} x_j} \cdot \mathbb{1}_{i \in \text{argmax}_{j \in N} x_j} & \text{(AUCTION)} \\
\frac{x}{\sum_{j \in N} x_j} & \text{(LOTtery)}
\end{cases}
\]

and

\[
C_i(x) = \begin{cases}
x_i & \text{(OWN bid)} \\
\min_{j \in N} x_j & \text{(LOWest bid)}
\end{cases}
\]

Before analyzing the equilibria of these mechanisms in detail, we provide a statement that will be helpful for the analysis of all mechanisms.

**Lemma 1.** Mechanisms with own-bid payment rule in an environment with public good element are strategically equivalent to mechanisms in an environment without public good element (i.e. \( \alpha = 0 \)) in which a prize of the amount \( \frac{V}{1-\alpha} \) is awarded. Mechanisms with lowest-bid payment rule in an environment with public good element are strategically equivalent to mechanisms in an environment with no public good element in which a prize of the amount \( \frac{V}{1-\alpha n} \) is awarded.

**Proof.** Let \( \tilde{\Pi}_i(x_i, x_{-i}) = \Pi_i(x_i, x_{-i}) - B \). In the own-bid mechanisms with public good element the payoff equals

\[
\tilde{\Pi}_i^{\text{OWN}}(\alpha, V)(x_i, x_{-i}) = -x_i + \alpha \cdot \sum_{j} x_j + P_i(x_i, x_{-i}) \cdot V
\]

\[
= \alpha \cdot \sum_{j \neq i} x_j + (1 - \alpha) \cdot [-x_i + P_i(x_i, x_{-i}) \cdot \frac{V}{1-\alpha}]
\]

\[
= \alpha \cdot \sum_{j \neq i} x_j + (1 - \alpha) \cdot \tilde{\Pi}_i^{\text{OWN}}(0, \frac{V}{1-\alpha})(x_i, x_{-i}),
\]
which is an affine transformation of the same setting without public good element and prize increased to \( \frac{V}{1-\alpha} \). In the lowest-bid mechanisms with public good element the payoff equals

\[
\tilde{\Pi}_i^{\text{LOW}}(\alpha, V)(x_i, x_{-i}) = \min_j \{ -\min_j x_j + P_i(x_i, x_{-i}) \cdot V \}
\]

\[
= (1 - \alpha n) \cdot \{ -\min_j x_j + P_i(x_i, x_{-i}) \cdot \frac{V}{1-\alpha n} \}
\]

which is an affine transformation of the setting without public good element and prize increase to \( \frac{V}{1-\alpha} \).

More informally, observe that for each dollar donated in the game we study, each player receives the fraction \( \alpha \) back in the form of benefits from the public good. Thus, donating a dollar in the setting with public good element is identical to donating only \( 1 - \alpha \) in the own-bid mechanisms and \( 1 - \alpha n \) in the lowest-bid mechanisms in the setting without a public good element. We can re-scale the payoffs by multiplying the prize and the individual donation by the appropriate factor and such an affine transformation of the payoffs does not change the equilibria of the game.

4 Optimal design

In this section we show that the lowest-bid all-pay auction is the optimal design. We first state some properties of symmetric mixed strategy equilibria and show that the expected payoff in a symmetric equilibrium is \( \frac{V}{1-\alpha} \). Next, we derive the cumulative distribution function of bids in a symmetric equilibrium and demonstrate that this symmetric equilibrium is unique. Then we show that a higher payoff cannot be generated by any voluntary participation mechanism. Finally, we show that asymmetric equilibria exist in the lowest-bid all-pay auction mechanism when the budget constraint is sufficiently high.

**Proposition 1** (Symmetric equilibrium in mixed strategies).

(a) No budget constraint. There is a unique symmetric equilibrium in mixed strategies with a cumulative distribution function given by the following differential equation

\[
F'(x) = \frac{1 - \alpha n}{V} \cdot \frac{(1 - F(x))^{n-1}}{(n-1)F^{n-2}(x)}
\]

with an initial condition \( F(0) = 0 \).

(b) High budget constraint: \( B \geq \frac{V}{n(1-\alpha n)} \). There is a unique symmetric equilibrium in which bidders randomize on the interval \([0, b]\) according to \( F(x) \) and donate their entire budget with a probability of \( 1 - F(b) \). The cutoff value \( b \) is determined by the unique solution to the equation

\[
\frac{V}{1-\alpha n} \cdot \frac{1 - F(b)^n}{n(1 - F(b))} = [1 - (1 - F(b))^{n-1}](B - b).
\]
(c) Low budget constraint: \( B < \frac{V}{n(1-\alpha n)} \). In the unique equilibrium bidders contribute their entire budgets.

Proof. Using Lemma 1, we focus on the environment without a public good. It is easy to see that no symmetric equilibrium in pure strategies exists because each bidder will have an incentive to increase her bid and secure the prize without paying more. We then consider the mixed strategy extension of the game in which each player \( i \in N \) chooses a cumulative distribution function \( F_i(x) \) over the set of pure strategies. Let also \( \phi_i(\tilde{x}) = F_i(\tilde{x}) - \lim_{x \uparrow \tilde{x}} F_i(x) \) denote the size of a mass point placed at bid \( \tilde{x} \) in the distribution. We proceed now in several steps.

**Step 1.** No mass points in the symmetric equilibrium distribution (except at the budget constraint \( B \)). Assume that there exists an atom in the symmetric equilibrium distribution, i.e. there is a mass point at bid \( \tilde{x} \). With a probability of \( \phi_i(\tilde{x}) \) there is a tie at this bid in which case bidder \( i \) wins the prize only with a probability of \( 1/n \). Consider a deviation according to which bidder \( i \) shifts the mass \( \phi_i(\tilde{x}) \) from \( \tilde{x} \) to \( \tilde{x} + \epsilon \). The total probability of winning the prize will increase by at least \((1-1/n) \cdot \phi_i^{n-1}(\tilde{x})\), while the payment will increase by no more than \( \epsilon \) (observe that the payment function is continuous). For a small enough \( \epsilon \) the deviation is profitable.

**Step 2.** The lower bound of the support of the symmetric equilibrium is zero. Assume to the contrary that the lower bound is \( \ell > 0 \). Because the distribution is atomless, the bid \( \ell \) asks bidder \( i \) to pay \( \ell \), but the chances of winning the prize remain zero. So, a bid of zero is a profitable deviation.

**Step 3.** In the symmetric equilibrium each bidder contributes on average the amount \( \frac{V}{n(1-\alpha n)} \).

As the bid of zero is in the support of the mixed strategy equilibrium, and the payoff is \( B \) when this bid is played, the expected payoff of each bidder in a symmetric equilibrium is \( B \). Let \( E \) be the expected payment of each bidder (assuming symmetric equilibrium strategies). The expected payoff of the equilibrium mixed strategy equals the expected payoff of each strategy in the support, in particular zero. Observe that with zero the chance of a bidder to win the prize is zero, and total contributions equal zero. Thus, if according to the symmetric equilibrium distribution function each bidder donates on average \( E \), the following equation holds for \( E \)

\[
\frac{V}{n} + \alpha \cdot n \cdot E - E = 0 \quad \iff \quad E = \frac{V}{n(1-\alpha n)}. 
\]

**Step 4.** Derivation of equilibrium mixed strategies. We consider the case without a budget constraint first. The equilibrium distribution function is atomless and the expected payoff in a symmetric equilibrium is zero with each bid \( x \). Hence, if we assume that the equilibrium distribution function is continuously differentiable, then it must satisfy the equation

\[
F^{n-1}(x) \cdot \frac{V}{1-\alpha n} - \int_0^x y \, d[1 - (1 - F(y))^{n-1}] - (1 - F(x))^{n-1} \cdot x = 0. 
\]
The first term is the expected gain from the winning the prize. The second term is the expected payment when at least one of the bidders bids below $x$. This is the lowest order statistics of all other bidders, conditional on this lowest order statistics being below $x$. The third term is the probability that all other bidders bid above $x$, times the payment $x$. The derivative of the left hand-side equals:

$$(n - 1) F^{n-2}(x) F'(x) \cdot \frac{V}{1 - \alpha n} - x \cdot \frac{d}{dx} [1 - (1 - F(x))^{n-1}]$$

$$= (n - 1) F^{n-2}(x) F'(x) \cdot \frac{V}{1 - \alpha n} - x \cdot (n - 1)(1 - F(x))^{n-2} F'(x)$$

$$= (n - 1) F^{n-2}(x) F'(x) \cdot \frac{V}{1 - \alpha n} - (1 - F(x))^{n-1}.$$  

So, the equilibrium distribution function must satisfy

$$F'(x) = \frac{1 - \alpha n}{V} \cdot \frac{(1 - F(x))^{n-1}}{(n - 1) F^{n-2}(x)}.$$  

**Step 5. The symmetric equilibrium is unique.** So far we derived one symmetric equilibrium. We will demonstrate here that this equilibrium is unique. We proceed by contradiction. Assume that there are two symmetric equilibria, given by the cumulative distribution functions $F(x)$ and $G(x)$. Using the standard definition of stochastic dominance, we will say that $F(x)$ first degree stochastically dominate $G(x)$ if $F(x) \leq G(x)$ with strict inequality for some $x$. Since both distribution are assumed to be equilibria, then it cannot be the case that $F(x)$ dominates $G(x)$ of first degree (or vice versa). Indeed, if this would be the case, then the lowest order statistics of $F(x)$, given by the distribution $1 - (1 - F(x))^n$ would stochastically dominate the lowest order statistics of $G(x)$, given by $1 - (1 - G(x))^n$. As expected revenue is $n$ times the expected value of the lowest order statistics, it is clear that both distributions will lead to different expected revenues. This contradicts to the result we established that the expected value in every mixed strategy equilibrium equals $\frac{V}{1 - \alpha n}$. As both functions, being equilibrium distributions, are continuous, and $F(0) = G(0) = 0$, then the two must cross at least one more time. Let us assume that $y > 0$ is the minimum point at which they cross again, i.e. $F(y) = G(y)$ and without loss of generality let us assume that $F(x) \leq G(x)$ for $0 \leq x \leq y$. Then, in the interval $[0, y]$, on average, because of the same stochastic dominance argument made before, one of the distributions will result in a higher payment for the bidders than the other. This is, however, not possible, because the expected payoff is zero at all points in the support, and at both points the probability of winning the item is the same. That is, the same increase in probability should be gained by the same increase in the expected payment. Thus, there is only one symmetric mixed strategy equilibrium.

**Step 6. Equilibrium derivation with the budget constraint $B$.** To show that the strategy profile described in the proposition is indeed an equilibrium, we need to show that all bids
that belong to the support yield the same expected revenue (of zero) for a bidder, given that the other bidders follow the described equilibrium strategy. Observe that all bids in \([0, b]\) generate the same probability of earning the prize as in the case with no budget constraint, and the expected payment when these bids are made are also the same. We will now show that the bid \(B\) generates the same payoff as the bid \(b\).

Note that the only case in which a bid \(B\) will win the item and a bid \(b\) will not is when there is at least one other bidder who bids \(B\). If several bidders bid \(B\) there will be a tie. Let bidder \(i\) bid \(B\). The additional probability for this bidder to win the item (compared to the situation in which she bids \(b\)) is given by the following binomial expression describing the probabilities of a tie:

\[
\sum_{j=0}^{n-1} \binom{n-1}{j} (1 - F(b))^j F(b)^{n-1-j} \frac{1}{j+1}.
\]

After a standard manipulation we obtain that the above expression equals

\[
\frac{1 - F(b)^n}{n(1 - F(b))}.
\]

Thus, the additional expected gain from bidding \(B\) instead of \(b\) is

\[
V = \frac{1 - F(b)^n}{1 - \alpha n} \cdot \frac{1 - F(b)^n}{n(1 - F(b))}.
\]

With a bid \(B\) bidder \(i\) pays the same as with the bid \(b\) when all other bidders bid below \(B\). When at least one other bidder bids \(B\), then bidder \(i\) pays \(B\). Thus, the additional cost of raising the bid to \(B\) is

\[
[1 - (1 - F(b))^{n-1}](B - b).
\]

The equation given in the proposition is that the two expressions above are equal. To see that bids in the interval \((b, B)\) do not lead to a higher payoff for a bidder observe that the winning probability and the expected payoff with these bids are the same as with the bid \(b\). The uniqueness of the symmetric equilibrium established in Step 5 guarantees that there is a unique \(b\) which solves the equation given in the proposition.

The cumulative distribution function can be given in explicit form for \(n = 2\) bidders. In the case of \(n > 2\) bidders the inverse of the cumulative distribution function can be given in explicit form. The results are summarized in the following proposition.

**Proposition 2** (Symmetric equilibrium distribution). In the case \(n = 2\) the symmetric equilibrium takes the form

\[
F(x) = 1 - e^{-c \cdot x}
\]

with \(c = \frac{1}{\alpha n}\).

In the case \(n > 2\) the inverse of the cumulative distribution function takes the form

\[
F(y)^{-1} = \frac{(n-1)}{c} \cdot \left[ \frac{1}{n-2} \left( \frac{y}{1-y} \right)^{n-2} - \frac{1}{n-3} \left( \frac{y}{1-y} \right)^{n-3} + \ldots - \frac{1}{2} \left( \frac{y}{1-y} \right)^2 + \frac{y}{1-y} - \ln(1 - y) \right]
\]
where \( n \) is odd, and
\[
F(y)^{-1} = \left( \frac{n-1}{c} \right) \cdot \left[ \frac{1}{n-2} \left( \frac{y}{1-y} \right)^{n-2} - \frac{1}{n-3} \left( \frac{y}{1-y} \right)^{n-3} + \ldots + \frac{1}{2} \left( \frac{y}{1-y} \right)^2 - \frac{y}{1-y} + \ln(1-y) \right]
\]
when \( n \) is even.

Proof. The equation is an autonomous equation. We denote \( y = F(x) \) and we can write
\[
\frac{dy}{dx} = c \cdot \frac{(1-y)^{n-1}}{(n-1)y^{n-2}}
\]
or
\[
\frac{(n-1)y^{n-2}}{c \cdot (1-y)^{n-1}} \cdot dy = dx,
\]
where \( c = \frac{1-\alpha n}{\alpha n} \). Integrating with \( z = \frac{y}{1-y} \) gives
\[
x + K = \left( \frac{n-1}{c} \right) \cdot \int \frac{y^{n-2}}{(1-y)^{n-1}} \, dy = \left( \frac{n-1}{c} \right) \cdot \int \frac{z^{n-2}}{z+1} \, dz.
\]
If \( n \) is odd we have
\[
x + K = \left( \frac{n-1}{c} \right) \cdot \int \frac{z^{n-2} - 1}{z+1} \, dz
= \left( \frac{n-1}{c} \right) \cdot \left[ z^{n-3} - z^{n-4} + \ldots + z + 1 - \frac{1}{z+1} \right] \, dz
= \left( \frac{n-1}{c} \right) \cdot \left[ \frac{1}{n-2} \left( \frac{y}{1-y} \right)^{n-2} - \frac{1}{n-3} \left( \frac{y}{1-y} \right)^{n-3} + \ldots + \frac{1}{2} \left( \frac{y}{1-y} \right)^2 - \frac{y}{1-y} + \ln(1-y) \right].
\]
If \( n \) is even we have
\[
x + K = \left( \frac{n-1}{c} \right) \cdot \int \frac{z^{n-2} - 1}{z+1} \, dz
= \left( \frac{n-1}{c} \right) \cdot \left[ z^{n-3} - z^{n-4} + \ldots + z - 1 + \frac{1}{z+1} \right] \, dz
= \left( \frac{n-1}{c} \right) \cdot \left[ \frac{1}{n-2} \left( \frac{y}{1-y} \right)^{n-2} - \frac{1}{n-3} \left( \frac{y}{1-y} \right)^{n-3} + \ldots + \frac{1}{2} \left( \frac{y}{1-y} \right)^2 - \frac{y}{1-y} + \ln(1-y) \right].
\]
In either case, the initial condition \( y(0) = 0 \) gives \( K = 0 \).

From Step 3 of the proof to Proposition 1 we can derive the following proposition.

**Proposition 3** (Expected revenue). When \( B \geq \frac{V}{n(1-\alpha n)} \), in the unique symmetric equilibrium, expected total donations equal \( \frac{V}{1-\alpha n} \). When \( B < \frac{V}{n(1-\alpha n)} \), donations aggregate to \( n \cdot B \) in equilibrium.

Our next result establishes the optimality of the lowest-price all-pay auction. In fact, this mechanism is optimal among a slightly larger set of mechanisms, including awarding multiple prizes of total value \( V \) and all possible ways in which the winner of each prize is determined.
Proposition 4 (Optimal mechanisms). Among all mechanisms which transfer a total value $V$ to the bidders and bidders participate voluntarily (i.e. bidders in equilibrium do not earn less compared to the case in which they don’t donate), the lowest-price all-pay auction generates the highest expected revenue of $\frac{V}{1-\alpha n}$.

Proof. Let us denote by $\varphi_i(F^*_i(x_i), F^*_i(x_{-i}))$ the transfer that the mechanism prescribes to bidder $i \in N$ in the (mixed strategy) Nash equilibrium of any voluntary mechanism. For the set of mechanisms in which case a single prize is awarded, we have $\varphi_i(F^*_i(x_i), F^*_i(x_{-i})) = P_i(F^*_i(x_i), F^*_i(x_{-i})) \cdot V$. Similarly, we denote by $C_i(F^*_i(x_i), F^*_i(x_{-i}))$ the expected payment of bidder $i$ in an equilibrium of a voluntary participation mechanism.

The idea of the proof is to establish an upper bound on the total revenue generated by any mechanism using the voluntary participation constraint. For each $i \in N$ this constraint is given by

$$\varphi_i(F^*_i(x_i), F^*_i(x_{-i})) + \alpha \cdot \sum_{j \in N} C_j(F^*_j(x_j), F^*_j(x_{-j})) \geq C_i(F^*_i(x_i), F^*_i(x_{-i})).$$

Summing over all $i$ we obtain

$$\sum_i \varphi_i(F^*_i(x_i), F^*_i(x_{-i})) + \alpha \cdot n \cdot \sum_{j \in N} C_j(F^*_j(x_j), F^*_j(x_{-j})) \geq \sum_i C_i(F^*_i(x_i), F^*_i(x_{-i}))$$

$$\iff V + \alpha \cdot n \cdot \sum_{j \in N} C_j(F^*_j(x_j), F^*_j(x_{-j})) \geq \sum_{j \in N} C_j(F^*_j(x_j), F^*_j(x_{-j}))$$

$$\iff \sum_{j \in N} C_j(F^*_j(x_j), F^*_j(x_{-j})) \leq \frac{V}{1-\alpha n}.$$

Thus, due to the participation constraint, no higher revenue can be generated by any voluntary mechanism.

We next turn to the discussion of asymmetric equilibria leading to zero expected payoff in the lowest-price all-pay auction.

Proposition 5 (Asymmetric equilibria). When $B \geq \frac{V}{n(1-\alpha n)}$ asymmetric equilibria exist. In every asymmetric equilibrium a zero contribution belongs to the support of at least one bidder, and the expected payoff of this bidder is zero. There are asymmetric equilibria in which at least one of the bidders does not contribute (with a probability of one) and the total amount of donations is zero in this equilibrium.

Proof. Assume, by way of contradiction, that zero is not in the support of any bidder. We will argue that all bidders must have the same lower bound of their support. Indeed, assume not, and take the bidder with the smallest lower bound of the support. Assume that this is bidder $i$ and the support is $\ell_i > 0$. Let the next lowest bound of the mixed strategy support be $\ell_j$, the lower bound for bidder $j$. Let $\phi_i([\ell_i, \ell_j])$ be the mass that bidder $i$ places on the interval $[\ell_i, \ell_j]$. As these bids never win (they always lose against bidder $j$), but require bidder $i$ to pay a positive amount, it will be a profitable deviation for bidder $i$ to transfer the mass
\[ \phi_i(\ell_i, \ell_j) \] to zero. Hence, we conclude that all bidders must have the same lower bound. Using arguments we established earlier regarding the symmetric equilibrium (see Step 1) we conclude that there is no mass point at the lower bound of the support of the mixed strategy equilibrium. Hence, by playing \( \ell_i > 0 \) a bidder will not win with a probability of one, but has to pay a positive amount. So, bidding zero will be a profitable deviation – a contradiction to the assumption that zero is not in the support of any bidder. It is easy to see that there are asymmetric equilibria in which all bidders bid \( B \) except for one who bids 0. This equilibrium play leads to zero total revenue.

5 The experiment

5.1 Design and procedures

In total our experiment consisted of twelve treatments, two for each mechanism. For each treatment an experimental session was conducted at xxx on separate days with different subjects. Subjects were recruited on campus to participate in an experiment in economic decision-making in which money can be earned. It was made clear that they would be paid in cash at the end of the session and that sessions take approximately 75 minutes. Subject were seated behind isolated computer terminals, via which the experiment was run, and acted in full anonymity (and without identity). After subjects read the instructions, answered the control questions correctly, and eventual clarifying questions were answered, the z-Tree software (Fischbacher, 2007) was started.

In each session there were 20 subjects participating (so, in total, 240 students participated in one of the treatments/sessions). In a series of 20 rounds these participants were randomly re-grouped in smaller groups of size four. The subjects were not aware of whom they were grouped with, but they did know that the group composition changed every round anew. Within each quadruple, a random chosen share of the subjects – the active bidders – were given a budget of 100 tokens. For each of the mechanisms we had one treatment where two subjects of each group were given a budget and one treatment where three of the four subjects were given a budget. The active bidders played the game as induced by the mechanism. The subjects that were not endowed with a budget – the passive bidders – could not win the prize and just benefitted from the voluntary contributions of the active bidders. For all treatments the marginal per capita return was 0.3 for all subjects (active and passive) and the prize was of size 20. After each round of play, subjects received information on all willingness-to-contributes, their contribution (in case the subject is an active bidder), the total contributions, whether they won the prize, and their own payoffs. In order to make sure that subjects take notice of the feedback carefully, they were asked to record part of the feedback on paper.

As a result of our distinction between active and passive bidders in combination with the size of the marginal per capita return contribution of the full budget is socially desirable.
on the aggregate whereas it is not desirable for the endowed active sub-society. The latter property is implemented as this results in the situation for which the equilibrium predictions are most interesting. To emphasize the difference between active and passive bidders, in addition to the rematching, ex post only one randomly chosen round is selected for payment – this excludes wealth effects in later rounds. The subjects earned on average $xxx for a session that took approximately xxx minutes.

5.2 Theoretical predictions and hypotheses

For budget equal to $B = 100$, marginal per capita return equal to $\alpha = 0.3$ and prize equal to $V = 20$, Table 1 presents the equilibrium predictions regarding contribution levels in the different mechanisms for number of bidders equal to $n = 2$ and $n = 3$. More precisely, it contains the expected individual willingness-to-contribution, the expected individual actual contribution and the expected group contribution as predicted according to theory and according to numerical computation. The theoretical result for the lowest-bid auction is presented in the previous section and that for the own-bid auction is provided in Appendix B. For the numerical computations we solved the game – in its discretized version as implemented in the experiment, but on a five times rougher grid – using the QRE-solver of Gambit that solves for the limiting logit equilibrium as introduced by McKelvey and Palfrey (1995).

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Number of bidders</th>
<th>Expected contributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>OWN–VCM</td>
<td>n = 2</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>OWN–LOT</td>
<td>n = 2</td>
<td>. . . 14.29 28.57</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>. . . 9.52 28.57</td>
</tr>
<tr>
<td>OWN–AUC</td>
<td>n = 2</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>LOW–VCM</td>
<td>n = 2</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>LOW–LOT</td>
<td>n = 2</td>
<td>. . . 33.55 19.85 39.70</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>. . . 85.06 64.54 193.62</td>
</tr>
<tr>
<td>LOW–AUC</td>
<td>n = 2</td>
<td>59.95 25.00 50.00 42.18</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>. 66.67 200.00 86.85</td>
</tr>
</tbody>
</table>

Table 1: Equilibrium predictions

From these predictions we derive the following testable hypotheses on the performance across and within mechanisms:

1. Addition of a competitive element via either all-pay auctions or lotteries increases voluntary contributions. Among the all-pay auction and lottery, the all-pay auction is the
more effective mechanism.

2. The lowest-bid payment rule generates more revenue than the own-bid payment rule.

3. The revenue is increasing in the number of (active) bidders in the lottery and the all-pay auction.

Notice that the third hypothesis is formulated on basis of the numerical prediction that takes into account the discretized domain of the bids. The theoretical solution for the continuous bidding domain only leads to this hypothesis for the lowest-bid payment rule; for the own-bid payment rule the revenue is independent of the number of bidders.

5.3 Results

As a result of the stranger matching with feedback on decisions after every round, we have just one truly independent observation per treatment. Despite of this, for each variable of interest, we construct units of observation that we treat as independent in our primary tests. For the individual willingness-to-contribute and actual contribution, we take for each individual her average willingness-to-contribute respectively actual contribution over all rounds (in case she was an active bidder) and treat the result as one observation. This provides us with 20 (dependent) observations for the individual willingness-to-contribute and actual contribution. For the group contribution, we take each group contribution in each round as one observation, which provides us with 100 observations. Table 2 presents the mean individual willingness-to-contribution and actual contribution and the mean group contribution in the different mechanisms for number of active bidders equal to $n = 2$ and $n = 3$.

Next, we run nonparametric tests to contest equality of group contribution levels between treatments. We reject the null hypothesis of equality when the $p$-value of the respective Mann-Whitney test is below 5%. We use the one-sided test outcome in case the equilibrium solution predicts inequality; otherwise we use the two-sided test.

First hypothesis (allocation rule). Opposed to the first part of the hypothesis, we find that the contribution raised via the own-bid lottery does not differ significantly from that of the standard voluntary contribution mechanism when there are three active bidders ($p = .247$). Moreover, although the treatment effect is significant for the own-bid all-pay auction ($p = .043$), the differential effect of the auction is negative when there are three active bidders. In presence of two active bidders, however, both mechanisms have an incremental impact on

---

6 Additional regressions to control for dependencies in spirit of Lange, List and Price (2007) have to be conducted.

7 Note that the level of dependence between observations is highest for the individual actual contribution in the treatments with lowest-bid payment rule.

8 Remark: the lowest-bid voluntary contribution mechanism is still to be run.

9 Two sessions for the lowest-bid payment rule are upcoming. Some of the interpretations below could be affected.
Table 2: Average contributions in the experimental sessions.

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Number of bidders</th>
<th>Mean contributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>OWN–VCM</td>
<td>n = 2</td>
<td>42.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>42.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>84.30</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>33.61</td>
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<td>33.61</td>
</tr>
<tr>
<td></td>
<td></td>
<td>99.70</td>
</tr>
<tr>
<td>OWN–LOT</td>
<td>n = 2</td>
<td>49.76</td>
</tr>
<tr>
<td></td>
<td></td>
<td>49.76</td>
</tr>
<tr>
<td></td>
<td></td>
<td>99.04</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>33.81</td>
</tr>
<tr>
<td></td>
<td></td>
<td>33.81</td>
</tr>
<tr>
<td></td>
<td></td>
<td>103.53</td>
</tr>
<tr>
<td>OWN–AUC</td>
<td>n = 2</td>
<td>57.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>57.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>118.68</td>
</tr>
<tr>
<td></td>
<td>n = 3</td>
<td>29.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>29.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>90.86</td>
</tr>
<tr>
<td>LOW–VCM</td>
<td>n = 2</td>
<td>70.14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>55.77</td>
</tr>
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<td></td>
<td>127.62</td>
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<tr>
<td>LOW–LOT</td>
<td>n = 2</td>
<td>59.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>40.20</td>
</tr>
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<td>20.00</td>
</tr>
<tr>
<td></td>
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<td>100.00</td>
</tr>
</tbody>
</table>

Our tests indicate that the contributions raised in auctions and lotteries differ significantly for all possible comparisons. Though, the treatment differences do not confirm the hypothesis that the auction is always the more effective mechanism. For the own-bid payment rule, the auction generates more revenue than the lottery when there are two active bidders ($p = .003$), but it generates less when there are three active bidders ($p = .008$). Precisely opposite, for the lowest-bid payment rule the auction outperforms the lottery when there are three active bidders ($p = .029$), while it is outperformed when there are two active bidders ($p = .001$).

**Second hypothesis (payment rule).** For the lottery, the lowest-bid payment rule yields significantly more revenue than the own-bid payment rule when there are two active bidders ($p = .012$), while there is no significant difference when there are three active bidders ($p = .327$). For the all-pay auction, the effect of lowest-bid payment rule on contributions is negative when there are two active bidders ($p = .000$), but positive when there are three bidders ($p = .002$).

**Third hypothesis (number of active bidders).** For the own-bid payment rule, we find that competition has a positive effect on total contributions in the voluntary contribution mechanism ($p = .011$), no effect in the lottery (one-sided: $p = .348$; two-sided: $p = .696$), and a negative effect in the auction ($p = .000$). For the lowest-bid payment rule, we find that competition has no effect in the lottery ($p = .464$) and a positive effect in the auction ($p = .000$). So, the introduction of the lowest-bid payment rule does not affect the differential.

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10See graphs in Appendix A.3.
effect of competition in the lottery, but changes a negative effect into a positive one for the all-pay auction.

**Optimality of the lowest-bid all-pay auction.** Our experimental data suggests that, in practice, the optimality of the lowest-bid all-pay auction hinges on the level of competition: only when there are sufficient active bidders, the lowest-bid all-pay auction is behaviorally optimal (among the mechanisms implemented). It appears that the differential effect of the level of competition (number of active bidders) only favors more competitive mechanisms (the auction) when risk is suppressed via the payment rule (paying the lowest-bid).

6 Conclusion

In an effort to bring budget deficits under control without raising taxes, communities increasingly rely on the non-profit sector to provide a variety of public goods and services which have traditionally been provided by governments. For instance, in her proposed budget for the fiscal years 2011–2013, the state of Washington governor Gregoire announced that “[…] the safety net will be stretched thin in some places and eliminated entirely in others.” Cuts in basic services such as, for instance, education and healthcare are on the agenda in virtually every state across the US. It is well-known that voluntary contribution mechanisms for the private provision of public goods are not be able to produce efficient outcomes because of free-riding incentives. In the past several decades economists have responded to the free-rider problem by studying, both theoretically and experimentally, various schemes from the private provision of public goods ranging from point provision mechanisms to matching and rebate schemes, to seed money and refunds. One commonly used but increasingly important source of revenue is the use of prize-based mechanisms, e.g. auctions and lotteries, for charitable fund-raising.

This paper derives the optimal fund-raising mechanism in a simple model in which a prize of commonly known value is awarded to one of the donors. Theoretically, the lowest-price all-pay auction is the optimal mechanism. Yet, this result is behaviorally fragile. We find that the lowest-price all-pay auction generates the highest revenue only if there is sufficient number of active bidders.

Our analysis opens various opportunities for further theoretical and experimental research. An assumption which is probably most often violated in reality is the symmetry of the bidders. In this paper, we assume symmetry across three important dimensions: budgets, individual benefit from the public good, and equilibrium behavior. Relaxing any of these symmetry assumptions, albeit theoretically challenging, can generate new insights. Recent work by Bos (2011) shows that the dominance of the own-bid all-pay auction over the lottery does not generally hold if bidders vary in the way they value the prize and the public good if this asymmetry is sufficiently strong. How to design optimal mechanisms when such asymmetries
Another important extension is to allow endogenous participation by donors. In a field experiment Carpenter et al. (2008) observes that, in contrast to theory, the winner-pay first-price auction generates more revenue than the own-bid all-pay auction (and the winner-pay second price auction) – a finding that the authors attribute to endogenous participation. In a more recent theoretical work Carpenter, Holmes and Matthews (2010) derive the symmetric Bayes-Nash equilibria in the above three mechanisms in a setting in which bidder have mechanism-specific entry costs and decide whether they wish to participate or not. The design of optimal fund-raising mechanisms with endogenous participation, however, is still unexplored even in symmetric settings.

Finally, our model considers the fund-raising activity in isolation of future fund-raisers. Many fund-raising efforts, are, however, repeated events in which behavioral spillovers play a role. Using a field experiment Landry, Lange, List, Price and Rupp (2010) find that previous donors are more likely to give than those who are asked for a first time to contribute, and explore the factors that keep donors committed to the cause. One important conclusion that Landry et al. (2010) draw is that donors initially attracted via economic mechanisms – such as auctions, lotteries, seed money, matching grants, etc. – are more likely to continue to contribute in the future than the ones attracted by “non-mechanism” factors (e.g. the appearance of the solicitor). Thus, theoretical and experimental work on optimal economic mechanisms for fund-raising that explicitly accounts for recurrence of fund-raising events presents an important avenue for future research.
References


A Theoretical predictions and experimental outcomes

A.1 Individual willingness-to-contribute

Figure 1: The figures provide the cumulative distribution over all individual decisions. Solid lines refer to theoretical prediction (Gambit with rougher discretization); dashed lines to experimental outcomes. Grey lines refer to two active bidders; black lines to three active bidders.
A.2 Individual actual contribution

![Graphs showing cumulative distribution over all individual actual contribution levels for VCM, Lottery, and Auction with Own-bid and Lowest-bid scenarios. Solid lines refer to theoretical prediction (Gambit with rougher discretization); dashed lines to experimental outcomes. Grey lines refer to two active bidders; black lines to three active bidders.](image)

Figure 2: The figures provide the cumulative distribution over *all* individual actual contribution levels. Solid lines refer to theoretical prediction (Gambit with rougher discretization); dashed lines to experimental outcomes. Grey lines refer to two active bidders; black lines to three active bidders.
A.3 Group contribution

Figure 3: The figures provide the cumulative distribution over all group contribution levels. Solid lines refer to theoretical prediction (Gambit with rougher discretization); dashed lines to experimental outcomes. Grey lines refer to two active bidders; black lines to three active bidders.
B The own-bid all-pay auction

In this appendix we present a new approach for the analysis of the own-bid all-pay auction. From Lemma 1 it follows that this auction has the same equilibria as an auction with a marginal per capita return of zero ($\alpha = 0$) in which a prize of size $\frac{V}{1-\alpha}$ is awarded to the winner. This property applies because the payoffs of the auction with an inflated prize are a linear transformation of the payoffs of the original auction for a public good we consider. Such a linear transformation of the payoffs leaves the best response correspondences unaltered, and, hence, has no effect on the set of equilibria. This observation allows us to use results from the recent literature on all-pay auction with complete information (Baye et al., 1996) and from the literature on caps on political lobbying (Che and Gale, 1998). We show that most of these results can be applied (with minor modifications) in the current setting, which allows us to describe all equilibria. In the case of two bidders there is a unique equilibrium given in symmetric mixed strategies. In the case of three or more bidders, we show that there is a unique symmetric (mixed strategy) equilibrium, and a continuum of asymmetric equilibria. In all equilibria, however, expected revenue is the same, does not depend on the number of bidders, and equals $\frac{V}{1-\alpha}$ provided that the budget constraint $B$ is not so small that all players prefer to donate their entire budget in equilibrium.

The next proposition summarizes the equilibria of the pay your own bid auction depending on the budget constraint $B$.

**Proposition 6 (Equilibrium strategies).**

(a) **Non-binding budget constraint:** $B \geq \frac{V}{1-\alpha}$. There are no Nash equilibria in pure strategies. In the case of two bidders there is a unique Nash equilibrium in mixed strategies. In this equilibrium bidders randomize on the interval $[0, \frac{V}{1-\alpha}]$ according to the following cumulative distribution function:

$$F(x) = \left( \frac{x}{\frac{V}{1-\alpha}} \right)^{\frac{1}{1-\alpha}}.$$

In the case of three or more bidders there is a unique symmetric Nash equilibrium with the c.d.f. $F(x)$ and a continuum of asymmetric equilibria. In the asymmetric equilibria some of the bidders do not contribute with a positive probability.

(b) **Binding budget constraint:** $\frac{V}{(1-\alpha)n} \leq B < \frac{V}{1-\alpha}$. There are no Nash equilibria in pure strategies. In the case of two bidders there is a unique Nash equilibrium. In this equilibrium bidders randomize on the interval $[0, 2B - \frac{V}{1-\alpha}]$ according to $F(x)$ and contribute their entire budget with a probability of $2(1 - (1 - \alpha)\frac{B}{V})$. In the case of three or more bidders there is a unique symmetric Nash equilibrium and a continuum of asymmetric equilibria. In the symmetric equilibrium bidders randomize on the interval $[0, b]$ according to $F(x)$ and donate their entire budget with a probability...
of $1 - F(b)$. The cutoff value $b$ is determined by the unique solution to the equation:

$$b \frac{n}{n-1} \cdot \frac{n}{n-1} - nB \cdot \frac{b}{n-1} + nB \cdot \left( \frac{V}{1-\alpha} \right)^{\frac{n}{n-1}} - \left( \frac{V}{1-\alpha} \right)^{\frac{n}{n-1}} = 0.$$  

In the asymmetric equilibria some of the bidders do not donate with a positive probability.

(c) Low budget constraint: $B < \frac{V}{(1-\alpha)n}$. There is a unique Nash equilibrium in which all participants contribute their entire budget $B$. The payoff of each participant is $V_n + (n\alpha - 1)B$.

Proof. Lemma 1 allows us to focus on the own-bid all-pay auction with a prize of $\frac{V}{1-\alpha}$. The uniqueness of equilibrium and the equilibrium distribution with a budget constraint in the case of two bidders is derived in Che and Gale (1998, Lemma 3 and Proposition 1). The general case of $n$ bidders with no budget constraint can be found in Baye et al. (1996). Here we will discuss the equilibria with a binding budget constraint of $B \leq \frac{V}{1-\alpha}$. The proof is organized as follows. We first show that the distribution function stated in the proposition indeed forms a symmetric equilibrium. Then we argue than no other symmetric equilibria exist. Finally, we discuss asymmetric equilibria.

Step 1. The stated distribution function forms a symmetric equilibrium. Observe that, when all other players play the stated mixed strategies, the payoff of bidder $i$ with all bids $x_i \in [0, b]$ equal $F(x_i)^{n-1} \cdot \frac{V}{1-\alpha} - x_i = 0$. We will now show that, for the given mixed strategies, with the bid $B$ each bidder earns the same payoff as with the bid $b$, which is zero. As bidder $i$ earns zero with all bids in $[0, b]$ this bidder will earn zero by contributing $B$ if and only if the expected payment of playing the equilibrium mixed strategy equals the bidder’s expected revenue that the mixed strategy brings, which is $\frac{V}{1-\alpha}n$ (in a symmetric mixed strategy the winning chances of all bidders are the same). This leads us to the following equation:

$$\int_0^b x \, dF(x) + (1 - F(b)) \cdot B = \frac{V}{n(1-\alpha)}.$$  

Observe that this equation has a unique solution because the left hand-side is increasing in $b$. As the left hand-side shows how much a bidder pays in expectation with a strategy with a variable $b$, higher $b$ means that a player transfers part of the mass placed on point $B$ to lower points. This obviously decreases the expected payoff. Solving the integral on the left hand-side and rearranging terms yields the equation for $b$ stated in the proposition. As all points in the stated equilibrium distribution yields for bidder $i$ the same payoff of 0, the stated mixed strategy yields the same payoff. It is easy to see that any bid in the interval $(b, B)$ yields a payoff smaller than zero because the winning chances of each bidder is the same as with the strategy $b$, but a bidder pays more. Thus, the described strategies are equilibrium strategies.

Step 2. No other symmetric equilibrium exists. We first observe that there are no mass points in the equilibrium distribution except possibly at $B$. The existence of mass points involves ties
at the mass point in which all bidders are involved (we consider symmetric mixed strategies). A bidder can marginally shift up the mass point, and this will be a profitable deviation for the bidder because the bidder will be able to resolve a tie which occurs with a strictly positive probability in his favor with marginal extra cost. Such a deviation is profitable (see also Hillman and Samet, 1987, footnote 7, for details). Next, we observe that zero is in the support of the distribution. Assume that the lower bound of the support is \( \ell_i > 0 \). As no mass points exist, the probability of winning the prize with this bid is zero, but this bidder pays a positive amount. Playing zero is hence a profitable deviation. Since zero is in the support, the expected payoff of a bidder with each bid in the support, in particular the equilibrium mixed strategy, must equal the expected payoff with zero. Let us assume that the average donation according to the equilibrium mixed strategy is \( E \). If all other bidders follow the equilibrium mixed strategy, the expected payoff with the equilibrium mixed strategy equals \( \frac{V}{n} + \alpha \cdot n \cdot E \). If the bidder bids zero, the bidder gains only from the donations of the remaining bidders to e private good, \( \alpha \cdot (n - 1) \cdot E \). Solving the equation

\[
\frac{V}{n} + \alpha \cdot n \cdot E = \alpha \cdot (n - 1) \cdot E,
\]

we obtain \( E = \frac{V}{(1 - \alpha)n} \), which is the expected donation of every symmetric mixed strategy equilibrium. The rest of the proof is identical to the uniqueness proof for the lowest-price all-pay auction (see Step 5 in the proof).

**Step 3. Asymmetric equilibria.** This auction format admits a variety of asymmetric equilibria. One class of asymmetric equilibria involve a number of \( m \), where \( n > m \geq 2 \) bidders being active and playing a symmetric mixed strategy as the one in a game with \( m \) bidders only, and the rest of the bidders submitting a bid of 0 with a probability of 1. It is straightforward that these strategy profiles are equilibria. Baye et al. (1996) present the set of all equilibria in the case of no budget constraint. All these equilibria remain equilibria with an appropriate modification of the equilibrium strategy in the interval \( [b, B] \) (these can be described in more detail). This completes Part (a) and Part (b) of the proposition.

We now move to the proof of Part (c). We first verify that the strategy profile in which all bidders donate their entire budget is an equilibrium, and then we demonstrate that no other equilibria exist. When all bidders play \( B \), the expected gain from earning the prize is \( \frac{V}{n(1-\alpha)} \) which exceeds \( B \). Any deviation of a single bidder to a lower amount will result in zero chance of earning the prize and is not profitable. Thus, all bidders contributing their entire amount is an equilibrium. We will next show that no other equilibria exist (in mixed or pure strategies).

We first verify that the lower bounds of the mixed strategy probability distribution of all bidders is the same: \( \ell_1 = \ell_2 = \cdots = \ell_n \). Assume, on the contrary that there exists a \( i \) such that \( \ell_i < \max_{j \neq i} \ell_j \), and observe that with the bids from the interval \( [\ell_i, \max_{j \neq i} \ell_j] \) bidder \( i \) always loses the contest to one of the other bidders. Then a deviation according which bidder \( i \) transfers the mass \( \phi_i(\ell_i, \max_{j \neq i} \ell_j) \) to \( B \) is profitable because the additional expected gain
from winning the prize is at least \( \phi_i[\ell_i, \max_{j \neq i} \ell_j] \cdot \frac{V}{(1-\alpha)n} \) and the additional expected cost is not higher than \( \phi_i[\ell_i, \max_{j \neq i} \ell_j] \cdot B \).

We now show that there is no mass point at the lower bound: \( \phi_i[\ell_i] = 0 \) for all \( i \). If there are two or more bidders with a positive mass on the point \( \ell_i \), each one of them (say bidder \( i \)) can improve his expected payoff by shifting the mass \( \phi_i[\ell_i] \) to the point \( \ell_i + \epsilon \). In doing so that bidder will for sure win against another player who plays \( \ell_i \). The expected gain from this deviation is the additional probability of winning times \( \frac{V}{1-\alpha} \). The expected cost is bounded from above by \( \epsilon \), and for \( \epsilon \) small enough the deviation is profitable. If bidder \( i \) is the only bidder with a mass point on \( \ell_i \), this bidder loses with the bid of \( \ell_i \) with certainty. Shifting the mass \( \phi_i[\ell_i] \) to the point \( B \) is profitable for this bidder. As we already know that all bidders have the same lower bound \( \ell_i \), and there is no mass point on that bound, we consider a deviation according to which bidder \( i \) shifts the mass \( \phi_i[\ell_i, \ell_i + \epsilon] \) to the point \( B \). For each \( \delta > 0 \) we can choose \( \epsilon \) small enough so that \( \prod_{j \neq i} \phi_j[\ell_i, \ell_i + \epsilon] < \delta \). Thus, by playing this deviation, bidder \( i \) increases his payoff from winning the prize by least \( \phi_i[\ell_i, \ell_i + \epsilon] \cdot (1/n - \delta) \frac{V}{1-\alpha} \). The additional cost of the deviation is no larger than \( \phi_i[\ell_i, \ell_i + \epsilon] \cdot B \). Thus, \( \delta \) can be chosen small enough so that the additional gain exceeds the cost of the deviation.

The own-bid all-pay auction format has three intriguing features. First, despite the multiplicity of equilibria in the case of \( n \geq 3 \) bidders, as we will establish in the next proposition, the expected revenue of all equilibria is the same. Second, the expected revenue is not affected by the budget constraint \( B \) unless \( B \) is so low relative to the prize \( V \) and the public good factor \( \alpha \) that bidders find it advantageous to contribute their entire budget. Finally, the expected revenue does not depend on the number of participants \( n \).

**Proposition 7 (Expected revenue).** When \( B > \frac{V}{n(1-\alpha)} \) all mixed strategy equilibria generate the same expected revenue of \( \frac{V}{1-\alpha} \) independent of the number of bidders \( n \). When \( B \leq \frac{V}{n(1-\alpha)} \) all participants contribute their entire budget \( B \) in equilibrium.

**Proof.** We already established that all the bids have the same lower bound at zero and that a person who bids zero has an expected payoff of zero (given that the other players are playing the equilibrium strategy). If there exists a bidder without a mass point on the lower bound, then the winning chances of another bidder who plays the lower bound is zero. In this case the lower bound can only be zero, and the payoff of all bidders in equilibrium is zero (otherwise the payoff would be negative, and bidders cannot get negative payoffs in equilibrium as otherwise they would bid zero). If all bidders have a mass point at the lower bound, then each bidder has an incentive to marginally increase the position of this mass point to avoid the tie with the other bidders. Thus, the only possibility is that the lower bound is zero and all bidders earn an expected payoff of zero at the lower bound. This shows that for all bidders the expected payoff in equilibrium is zero as this is the payoff of any strategy in the support. As the prize is assigned with a probability of one, it follows that the expected donations of all bidders equal \( \frac{V}{1-\alpha} \).